

Vector meson dominance and the π^0 transition form factor

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It is shown that the π^0 transition form factor $F(Q_1^2, Q_2^2)$ differs substantially from its one-real-photon limit $F(Q_1^2, 0)$ even for rather small values of Q_2^2 ($\approx 0.1 \text{ GeV}^2$), which cannot be excluded in experiments with one “untagged” electron. It indicates that the comparison of data with theoretical calculations, which usually assume $Q_2^2 = 0$, may be untrustworthy. Our phenomenological model of the π^0 transition form factor is based on the vector-meson-dominance hypothesis and all its parameters are fixed by using the experimental data on the decays of vector mesons. The model soundness is checked in the two-real-photon limit, where it provides a good parameter-free description of the $\pi^0 \rightarrow 2\gamma$ decay rate, and in the π^0 Dalitz decay. The dependence of $F(Q_1^2, Q_2^2)$ on Q_1^2 at several fixed values of Q_2^2 is presented and the comparison with existing data performed.

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The issue of the π^0 transition form factor has recently attracted renewed interest in connection with the precise measurements of the *BABAR* Collaboration [1], which seem to indicate that the asymptotic limit predicted by perturbative QCD [2] has been exceeded. A comprehensive review of the current theoretical situation with an extended list of references can be found in Ref. [3]. On a phenomenological side, it has recently been shown [4] that the vector-meson-dominance (VMD) hypothesis [5–7] leads to a correct description of the two-photon decay of the π^0 if the parameters are fixed by the data on the partial decay widths of the vector mesons [8]. The rate of this decay is related to the real-photon limit $F(0,0)$ of the π^0 transition form factor. It is therefore tempting to use the VMD also for the construction of the π^0 transition form factor, which parametrizes the dynamics of the process in which two off-mass-shell photons fuse and form a π^0 .

The experimental data on the π^0 transition form factor are taken in the process $e^- + e^-(e^+) \rightarrow e^- + \pi^0 + e^-(e^+)$, where the photon virtualities are given by the electron (e^+ or e^-) momenta transfer squared, $Q_i^2 = -q_i^2$, $i = 1, 2$. To get the one-real-photon transition form factor only the data sample is utilized in which one lepton exhibits a small momentum transfer, e.g., $|q^2| < 0.18 \text{ GeV}^2$ in the latest *BABAR* experiment [1].

The transition form factor $F(Q_1^2, Q_2^2)$ is defined by its appearance in the $\gamma^* \gamma^* \pi^0$ vertex, which describes the fusion of two virtual photons with the four-momenta q_1 and q_2 into a π^0

$$T_{\mu\nu} = -ie^2 \epsilon_{\mu\nu\alpha\beta} q_1^\alpha q_2^\beta F(Q_1^2, Q_2^2), \quad (1)$$

where e is the elementary electric charge. This definition of the transition form factor agrees with that used in [1, 9], but differs from that in [10], where the factor e^2 was absorbed in $F(Q_1^2, Q_2^2)$. The two-real-photon value of the pion transition form factor is related to the two-

photon decay width of the π^0 by

$$\Gamma(\pi^0 \rightarrow 2\gamma) = \frac{\pi\alpha^2}{4} m_\pi^3 F^2(0,0), \quad (2)$$

where α is the fine-structure constant.

Another process in which the transition form factor plays a role is the decay $\pi^0 \rightarrow e^+ e^- \gamma$, which was suggested by Dalitz [11] as an explanation of the anomalous events recorded in photographic emulsions exposed in high-flying balloons [12]. The original evaluation of the branching ratio

$$B = \frac{\Gamma(\pi^0 \rightarrow e^+ e^- \gamma)}{\Gamma(\pi^0 \rightarrow 2\gamma)} \quad (3)$$

by Dalitz as well as a later one [13], did not consider a possible form factor. The latter was included in [14]. The differential branching ratio (3) in the Berman–Geffen [14] variable $x = M^2/m_\pi^2$, where M is the mass of the $e^+ e^-$ pair, reads as

$$\frac{dB}{dx} = \frac{2\alpha}{3\pi} \frac{(1-x)^3}{x} \left(1 + \frac{2\epsilon}{x}\right) \sqrt{1 - \frac{4\epsilon}{x}} F_D^2(x), \quad (4)$$

where $\epsilon = m_e^2/m_\pi^2$. Form factor F_D is related to the π^0 transition form factor by

$$F_D(x) = \frac{F(-xm_\pi^2, 0)}{F(0,0)}. \quad (5)$$

The total branching ratio (3) is not very sensitive to the shape of the form factor (5) and cannot serve as a stringent test of theoretical calculations or phenomenological models. Even the original Dalitz formula, which did not include the form factor at all, leads to $B = 1.185\%$, which agrees with the experimental value of $(1.188 \pm 0.035)\%$ [15]. The form factor (5) is at small x usually parametrized as

$$F_D(x) = 1 + ax.$$

In 1961, Gell-Mann and Zachariasen [6] showed that the form factor F_D is dominated by two resonances, namely ρ and ω , and got a positive a equal to $m_\pi^2(m_\rho^{-2} + m_\omega^{-2})/2$, in agreement with today's observations. Until late 1980's, most experiments had indicated negative values of a , see [16] and a list of preceding experimental results there.

Our model of the pion transition form factor $F(Q_1^2, Q_2^2)$ is defined in Fig. 1. In addition to the $\rho\omega$ intermediate state considered in [4, 6, 7], we include the following companions of the ρ^0 : ϕ , $\omega(1420)$ (denoted as ω' in what follows), and $\omega(1650)$ (denoted as ω''). All these resonances, except ω'' , have recently been considered in the model of the one-real-photon transition form factors of π^0 , η , and η' [17].

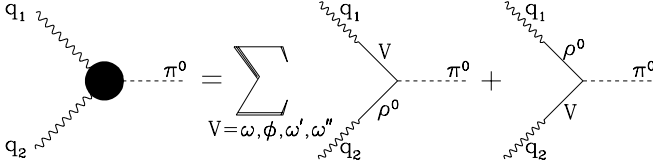


FIG. 1. Feynman diagrams defining our model of the π^0 transition form factor.

The combination of an isoscalar resonance with an isovector one in the intermediate states is unique for the π^0 transition form factor and implies the equal contribution of the $I = 0$ and $I = 1$ currents to $F(0, 0)$. With the η and η' , the situation is different.

In order to evaluate the Feynman diagrams depicted in Fig. 1, we use the Lagrangian

$$\mathcal{L}_{V\rho\pi} = G_{V\rho\pi} \epsilon_{\mu\nu\alpha\beta} (\partial^\mu V^\nu) (\phi \cdot \partial^\alpha \rho^\beta), \quad (6)$$

where V is the operator of a neutral vector meson field, ρ is the isovector of the ρ meson fields, and ϕ is that of the pion fields. The coupling of photons to neutral vector mesons is given by the VMD Lagrangian (a little nonstandard notation of [18] adopted from [19] is used)

$$\mathcal{L}_{\text{em}} = - \sum_V \frac{e g_{V\gamma}}{2} m_V^2 A^\mu V_\mu, \quad (7)$$

where A^μ is the electromagnetic field operator.

After comparing the amplitude corresponding to Feynman diagrams in Fig. 1 with the definition of the transition form factor (1) we extract the latter in the following form

$$F(Q_1^2, Q_2^2) = \frac{g_{\rho\gamma}}{4} \sum_{V=\omega, \phi, \dots} G_V [R_V(Q_1^2) R_\rho(Q_2^2) + R_V(Q_2^2) R_\rho(Q_1^2)], \quad (8)$$

where $G_V = G_{V\rho\pi} g_{V\gamma}$ and functions

$$R_V(Q^2) = \frac{m_V^2}{m_V^2 + Q^2} \quad (9)$$

are the scalar parts of the vector meson propagators below the physical cut threshold in $s = -Q^2$, $s_{\text{th}} = m_\pi^2$.

Tensor parts do not contribute thanks to the presence of the Levi-Civita tensor in (6). For $R_\rho(Q^2)$ we will alternatively use the form

$$R_\rho(Q^2) = \frac{M_\rho^2(0)}{M_\rho^2(-Q^2) + Q^2} \quad (10)$$

with the running mass squared given by the dispersion formula

$$M_\rho^2(s) = M_\rho^2(0) - \frac{s}{\pi} \mathcal{P} \int_{m_\pi^2}^{\infty} \frac{m_\rho \Gamma_\rho(s')}{s'(s' - s)} ds' \quad (11)$$

and satisfying $M_\rho^2(m_\rho^2) = m_\rho^2$ and $dM_\rho^2/ds = 0$ at $s = m_\rho^2$. The energy dependent total width Γ_ρ includes the contributions from the following final states: $\pi^+\pi^-$, $K\bar{K}$, $\pi^0\omega$, $\eta\pi\pi$, $\pi^0\gamma$, $\eta\gamma$, and $\pi^+\pi^-\gamma$. For details, see [20].

Inserting (8) into (5) and calculating the derivative of $F_D(x)$ at $x = 0$ we are getting the following expression for the slope of the Dalitz decay form factor

$$a = \frac{m_\pi^2}{2} \left[\frac{1}{M_\rho^2(0)} (1 - D) + \frac{1}{\sum_V G_V} \sum_V \frac{G_V}{m_V^2} \right]. \quad (12)$$

Here, D is the derivative of the running mass (11) at $s = 0$. When a fixed mass form (9) is used also for $R_\rho(Q^2)$, then $D = 0$ and $M_\rho^2(0) = m_\rho^2$. If we, in addition, keep only the first term $V = \omega$ in the sums over V above, we recover the Gell-Mann-Zachariasen [6] formula.

The values of coupling constants $G_{V\rho\pi}$ are determined as follows. For the vector mesons with masses below or close to the $\rho\pi$ threshold [$\omega(782)$, $\phi(1020)$] we consider the decay chain $V \rightarrow \rho + \pi \rightarrow 3\pi$. The comparison of the decay width formula given in [18] with the recommended value from [15] yields the product $G_{V\rho\pi}^2 g_\rho^2$, where g_ρ is the coupling constant of the usual $\rho\pi\pi$ interaction Lagrangian. Using the value $g_\rho^2 = 35.70 \pm 0.19$, as it follows from the ρ meson decay width [15], we get $G_{V\rho\pi}^2$. For the vector mesons with higher masses [ω' , ω'' , $J/\psi(1S)$] it is sufficient to explore the simpler $V \rightarrow \rho + \pi$ decay width formula. Concerning the ω' , the Review of Particle Physics [15] gives only intervals for its mass and total width and no quantitative estimate for the $\rho\pi$ branching fraction. We use therefore the values [21] $m_{\omega'} = (1.38 \pm 0.02 \pm 0.07) \text{ GeV}/c^2$ and $\Gamma_{\omega'} = (0.13 \pm 0.05 \pm 0.10) \text{ GeV}$ as measured by the BABAR Collaboration [22] and $B(\omega' \rightarrow \rho + \pi) = (69.9 \pm 2.9)\%$ from the wavelet analysis [23] of the e^+e^- annihilation data. In the case of the ω'' , we again use the BABAR [22] values $m_{\omega''} = (1.667 \pm 0.013 \pm 0.006) \text{ GeV}/c^2$ and $\Gamma_{\omega''} = (0.222 \pm 0.025 \pm 0.020) \text{ GeV}$ with the wavelet analysis [23] branching fraction $B(\omega'' \rightarrow \rho + \pi) = (38.0 \pm 1.4)\%$. The resulting values of the coupling constants $G_{V\rho\pi}$ squared are summarized in Table I. The coupling constants $G_{\omega\rho\pi}$ and $G_{\phi\rho\pi}$ themselves differ in sign, as it follows from the analysis of the $\rho \rightarrow \pi\gamma$ decay [18] and from the SU(3) symmetry [19]. This results in the negative sign of G_ϕ shown in Table I.

TABLE I. The squares of the coupling constants in Lagrangians (6) and (7) obtained from the vector meson decay data described in the text. Also shown are the parameters G_V , which enter the transition form factor (8).

V	$G_{V\rho\pi}^2$ (GeV $^{-2}$)	$g_{V\gamma}^2 \times 10^2$	G_V (GeV $^{-1}$)
$\omega(782)$	216.2 ± 3.0	1.375 ± 0.046	1.724 ± 0.031
$\phi(1020)$	0.676 ± 0.020	2.214 ± 0.031	-0.122 ± 0.002
$\omega(1420)$	11.7 ± 1.1	0.20 ± 0.17	0.152 ± 0.136
$\omega(1650)$	3.97 ± 0.61	0.76 ± 0.11	0.174 ± 0.026

Now, we determine the coupling constants in the VMD Lagrangian (7). The value $g_{\rho\gamma}^2 = 4/g_\rho^2 = 0.1120(10)$ follows from the normalization of the charged pion form factor. The squares of other coupling constants $g_{V\gamma}$ are evaluated from the dilepton decay width of the corresponding vector mesons. In the case of the ω , the e^+e^- decay width from [15] is used. For the ϕ , we obtain it as a product of the full width and the e^+e^- branching fraction. To get $g_{\omega'\gamma}^2$, we combine the full width from [22] (shown above) with the e^+e^- branching fraction $B(\omega' \rightarrow e^+e^-) = (32 \pm 1) \times 10^{-7}$ from [23]. Let us note that the product of the $\rho\pi$ and e^+e^- branching fractions from [23] agrees with $B(\omega'' \rightarrow e^+e^-)B(\omega'' \rightarrow \rho + \pi) = (1.3 \pm 0.1 \pm 0.1) \times 10^{-6}$ got later by the *BABAR* Collaboration [24]. The same concerns the full width. Unfortunately, the situation with the ω' is more controversial. The product of the $\rho\pi$ and e^+e^- branching fractions from the wavelet analysis [23] is about twofold of that observed by *BABAR* [24]. In order not to overestimate the contribution of the ω' to the π^0 transition form factor, we use the *BABAR* value $B(\omega' \rightarrow e^+e^-)B(\omega' \rightarrow \rho + \pi) = (0.82 \pm 0.05 \pm 0.06) \times 10^{-6}$ to get $G_{\omega'}^2 = G_{\omega'\rho\pi}^2 g_{\omega'\gamma}^2$, from which the estimate of $g_{\omega'\gamma}^2$ presented in Table I is obtained.

As all the parameters of our model of the π^0 transition form factor are now determined, we are ready to test its soundness by evaluating the characteristics of the two-photon and Dalitz decays of the neutral pion and comparing them to the measured values. To avoid the cumulation of the errors, we do not use Table I for the error analysis, but calculate the errors of the results directly from the input data (masses, full widths, branching fractions, and their errors). To account for possible correlations among the input quantities, we sum the contributions to the final errors from various sources linearly. Two of the calculated quantities, namely the mean lifetime of π^0 and the Dalitz decay slope parameter are shown in Table II for various versions of our model, i.e., for various choices of the isoscalar vector mesons entering the sum in Eq. (8) and for two possible treatments of the ρ propagator. The branching ratio of the Dalitz decay to the two-photon decay is not shown in the table. It acquires the same value of 1.196(1)% in all versions of the model, in agreement with the experimental value of 1.188(35)% [15]. Our results concerning the π^0 transition form factor are presented in Figs. 2 and 3. The

TABLE II. Comparison of various versions of our model with the π^0 decay data. When converting the calculated $\pi^0 \rightarrow 2\gamma$ decay rate into the π^0 mean lifetime τ , the experimental branching fraction of 0.988 was used. The results above the horizontal divider were obtained with the constant ρ mass, those below it with the running mass (11).

V	$\tau \times 10^{17}$ (s)	$a \times 10^2$
ω	7.6 ± 0.4	3.002 ± 0.002
ω, ϕ	8.8 ± 0.6	3.049 ± 0.004
ω, ϕ	8.8 ± 0.6	3.282 ± 0.003
ω, ϕ, ω'	7.3 ± 1.9	3.190 ± 0.097
ω, ϕ, ω''	7.2 ± 0.7	3.163 ± 0.024
$\omega, \phi, \omega', \omega''$	6.1 ± 1.7	3.089 ± 0.099
Data [15]	8.4 ± 0.5	3.2 ± 0.4

contributions of the various $\pi^0 \rho^0 V$ vertexes to the one-real-, one-virtual-photon form factor together with their sum are shown in Fig. 2. They were all calculated using the running mass (11) of the ρ meson. In addition, the sum of the same contributions but calculated using the fixed ρ mass is depicted by a dotted curve. It differs from the running-mass case only marginally. The strong

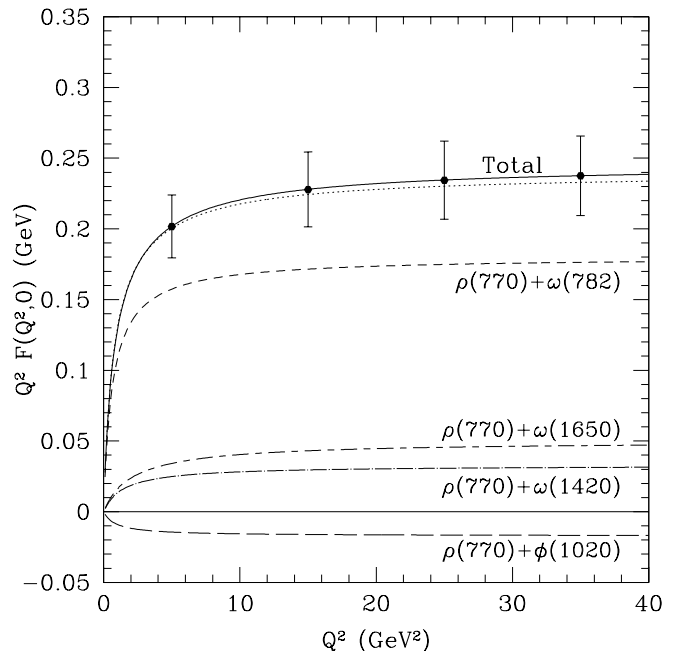


FIG. 2. The individual contributions to the pion transition form factor $F(Q^2, 0)$ multiplied by Q^2 and their sum (solid curve) with the uncertainties originating in the errors of input parameters. The sum of individual contributions (not shown) that were calculated assuming the constant mass of the $\rho(770)$ is depicted by the dotted curve.

dependence of the transition form factor $F(Q_1^2, Q_2^2)$ on the virtuality Q_2^2 , which is given by the four-momentum transfer of the “untagged” electron, is demonstrated in Fig. 3. The expression $Q_1^2 F(Q_1^2, Q_2^2)$ is presented as a

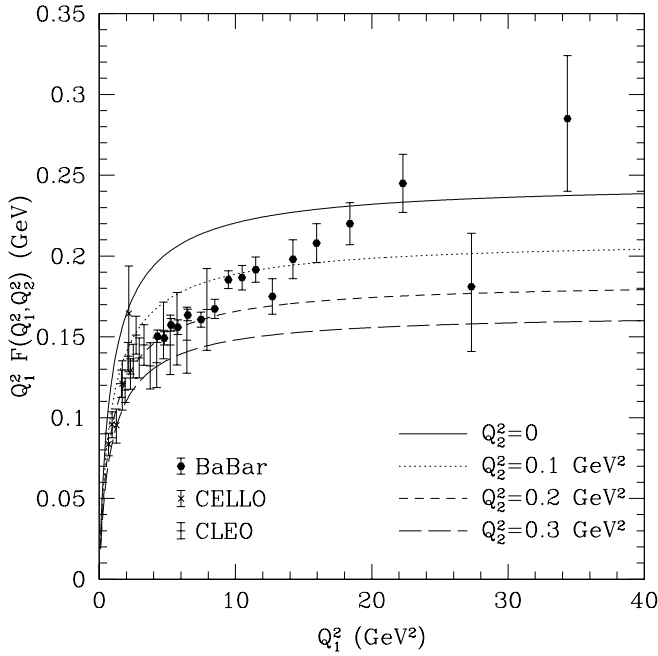


FIG. 3. Dependence of $Q_1^2 F(Q_1^2, Q_2^2)$ on Q_1^2 for four fixed values of Q_2^2 calculated in the full version of our model. The CELLO [10], CLEO [9], and BABAR [1] data are also shown.

function of Q_1^2 for four different virtualities Q_2^2 from 0 to 0.3 GeV^2 . The full version ($V = \omega, \phi, \omega'$, and ω'') of our model with running mass of the ρ is used. Comparison with existing data indicates that our model would be able to describe them if the (unmeasured, but certainly nonvanishing) absolute value of the momentum transfer squared of the “untagged” electron decreased with the rising Q_1^2 . A more quantitative account is given in Table III, which shows that the agreement of our model with the data below $Q_1^2 = 9 \text{ GeV}^2$ is excellent but deteriorating for higher Q_1^2 . There are two possible ways of improving our model in an attempt to get a better description of the high- Q_1^2 data. First, the inclusion of higher isovector

$[\rho(1450), \rho(1700)]$ and isoscalar $[\phi(1680), \phi(2170)]$ resonances. Second, a consequent use of the running-mass propagators also for other resonances [not only for $\rho(770)$ as here], which guarantee the correct analytic properties of the transition form factor.

TABLE III. The comparison of our parameter-free model with the data in various Q_1^2 ranges. The virtuality Q_2^2 of the photon radiated from the “untagged” electron was assumed the same for all n data points within a particular Q_1^2 range and determined by minimizing the χ^2 . The confidence levels (C.L.) are also shown.

$Q_1^2 \text{ (GeV}^2\text{)}$	n	χ^2	$Q_2^2 \text{ (GeV}^2\text{)}$	C.L. (%)
0–9	27	10.67	0.187 ± 0.008	99.7
9–18	6	3.77	0.107 ± 0.013	58.3
18–36	4	4.47	0.010 ± 0.025	21.5

The effect of nonvanishing Q_2^2 has already been quantitatively studied in terms of the photon momentum asymmetry parameter $A = (Q_1^2 - Q_2^2)/(Q_1^2 + Q_2^2)$ within the spectral quark model [25] and the incomplete vector-meson dominance (IVMD) [3]. In the former, the transition form factor decreases with rising Q_2^2 (falling A)—see Fig. 2 in [25]—as in our model. In the latter, the tendency is inverse—see Fig. 7 in [3]. This difference is caused by not keeping the IVMD parameters c and M_V constant, but allowing them to vary in order to get the best fit for each particular A .

Our model, in spite of its deficiencies, supports the conclusion of Refs. [3, 25] that it is important to pay more attention to the dependence of the transition form factor on both virtualities in theoretical calculations and experimental analyses.

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